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Aircraft Cruise-Dash Optimization
Periodic versus Steady-State Solutions *

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AIRCRAFT CRUISE-DASH OPTIMIZATION:

PERIODIC VERSUS STEADY-STATE SOLUTIONS

Abstract

This paper conducts a comparitive study of periodic and steady-state solutions for aircraft cruise-dash optimization. The solutions are in the point-mass model. The cost functional is an average weighted sum of the fuel used and the time taken. Previous work on cruise has determined that the steady-state solution fails a Jacobi-type test, conducted in frequency domain. Periodic solutions have been obtained for the same problem that use lesser fuel. The periodic solutions have been shown to be locally optimal. Similar analysis is carried out in the current effort for the cruise-dash problems that have non-zero weights on the time taken. As the weight on the time is increased, the difference in the costs become less and less. For all values for the weight on the time above a certain value, the steady-state solutions <u>are</u> locally optimal. The structure of the periodic solutions become intricate. The periodic solutions seem to "approach" the steady-state solution as the weight on time is increased.

List of Symbols

C_D.................Drag coefficient

C_L.................Lift coefficient

f....................Right hand side of vector state equation

g....................Acceleration due to gravity (m/s/s)

H .... Pseudo-Hamiltonion

h .... Altitude (m)

j .... Imaginary number, $\sqrt{-1}$

J .... Cost functional (N/m)

L .... Lift (N)

Q .... Maximum fuel flow rate (N/s)

T .... Maximum thrust (N)

u .... Vector of controls

V .... Airspeed (m/s)

W .... Aircraft weight (N)

W_f .... Fuel used (N)

X .... Range (m)

X_f .... Final range, wavelength (m)

z .... Vector of states

Greek Symbols.

δ .... Variation

η .... Throttle setting

γ .... Flight path-angle (rad.)

λ .... Co-state vector

λ_x .... Co-state variable

Π .... Frequency test matrix

θ .... Weight in cost functional (N/s)

ρ .... Air density (kg/cu.m)

σ .... TSFC (N/N/s)

ω..................Frequency (rad/s)

Introduction

In a historic paper, Speyer [Speyer] showed that for the cruise problem, the steady-state solution fails a Jacobi-type test conducted in frequency domain. For a band of frequencies, the second variation of the cost is negative, indicating that certain oscillatory controls can achieve lower costs. These results are valid only for trajectories in the vicinity of the steady-state solutions. Later investigations revealed that periodic solutions in the point-mass model could, indeed, achieve lower costs [Dannemil,Hsuang, Grimm]. An important point to be made here is that the frequency of periodic solutions is exactly as prophesied by the frequency test [Thesis]. This is at one end of the cruise-dash spectrum. At the other end, for the dash or minimum-time problem, it is known that the optimal trajectory is a full-throttle steady-state solution. So, somewhere in the cruise-dash regime, the steady-state solution becomes locally optimal and the periodic solutions lose their optimality. This paper investigates this behavior.

Steady-State Solutions:

A time invariant control is the simplest control that can be implemented. This is the main reason for interest in a steady-state solution. Often the simplicity of implementation supersedes any other requirement and the constant controls are chosen over dynamically varying controls, even if it results in a non-optimal operation of the aircraft.

The optimal control problem is to

$$\text{Min } J = \frac{1}{X_f} \int_0^{X_f} \frac{\eta Q + \theta}{V \cos \gamma} dx \tag{1}$$

subject to

$$\begin{aligned} V' &= \frac{g}{V \cos \gamma}[P - \sin \gamma] \\ \gamma' &= \frac{g}{V^2}\left[\frac{L}{W \cos \gamma} - 1\right] \\ h' &= \tan \gamma \end{aligned} \tag{2}$$

The steady-state solution requires that all the state-rates be zero. Hence,

$$\begin{aligned} V' &= \frac{g}{V \cos \gamma}[P - \sin \gamma] = 0 \\ \gamma' &= \frac{g}{V^2}\left[\frac{L}{W \cos \gamma} - 1\right] = 0 \\ h' &= \tan \gamma = 0 \end{aligned} \tag{3}$$

From these equations, it is readily seen that the solution must satisfy

$$\begin{aligned} \gamma &= 0 \\ L &= W \quad \text{which gives } C_L \\ \eta &= \frac{D}{T} \end{aligned} \tag{4}$$

Note that the drag is evaluated at lift equals weight. The altitude and velocity are then chosen to minimize the cost functional with these constraints. Then

$$\frac{\partial J}{\partial V} = 0$$

$$\frac{\partial J}{\partial h} = 0 \qquad (5)$$

As all the states are constant, the cost functional reduces to

$$J_{ss} = \frac{\eta Q + \theta}{V} \qquad (6)$$

The final range, X_f is of no consequence. The conditions on the altitude and velocity now become:

$$\frac{\partial}{\partial V}\left[\frac{\eta Q + \theta}{V}\right] = 0$$

$$\frac{\partial}{\partial h}\left[\frac{\eta Q + \theta}{V}\right] = 0 \qquad (7)$$

subject to $L = W$ and $\eta T = D$

The steady-state solution is also a singular point in the state-costate space. This implies that the rates of states and co-states are zero. Equations (5.1) hold. In addition,

$$\lambda_V{}' = -\frac{\partial H}{\partial V} = 0$$

$$\lambda_\gamma{}' = -\frac{\partial H}{\partial \gamma} = 0 \qquad (8)$$

$$\lambda_h{}' = -\frac{\partial H}{\partial h} = 0$$

where,

$$H = \frac{\eta Q + \theta}{V \cos \gamma} + \lambda_V \frac{g}{V \cos \gamma}\left[\frac{P}{W} - \sin \gamma\right] +$$
$$\lambda_\gamma \frac{g}{V^2}\left[\frac{L}{W \cos \gamma} - 1\right] + \lambda_h \tan \gamma \tag{9}$$

In addition, the controls already obtained in (5.2) must satisfy the Pontryagin minimum principle. Note that the throttle appears linearly in the state equations. The throttle position is generally within the bounds and the control is singular. The partial of the Hamiltonian with respect to the throttle must be zero. So,

$$\frac{\partial H}{\partial C_L} = 0 \quad \text{and} \quad \frac{\partial H}{\partial \eta} = 0 \tag{10}$$

From $\frac{\partial H}{\partial \eta} = 0$, one obtains

$$\lambda_V = -\frac{\sigma W}{g} \tag{11}$$

From $\frac{\partial H}{\partial C_L} = 0$, one obtains

$$\lambda_\gamma = \lambda_V V \frac{\partial C_D}{\partial C_L} \tag{12}$$

From $\lambda_\gamma' = 0$,

$$\lambda_h V = -\lambda_V g \tag{13}$$

Thus, at steady-state, $\gamma = 0$ and C_L, η, λ_V, λ_γ, λ_h are known functions of altitude and velocity. So only the altitude-velocity combination has to determined using the following equations:

$$\lambda_V' = -\frac{\partial H}{\partial V} = 0$$
$$\lambda_h' = -\frac{\partial H}{\partial h} = 0 \tag{14}$$

These are fairly well behaved functions and numerical solutions are easily obtained. Figure 1 shows the average costs for steady-state as a function of the cost weighting parameter, θ. Figure 2 shows the average costs against the average velocity. As expected, as the average velocity goes up, the costs increase. Next, the optimality of these solutions is considered.

The Frequency Test

Ref.[Speyer,1976] gives a second order Jacobi-type test for the optimality of the steady-state solutions. The problem is:

$$\operatorname*{Min}_{u} \frac{1}{X_f} \int_0^{X_f} L dX \tag{15}$$

subject to the state equations:

$$\dot{z} = f(z, u, t) \tag{16}$$

The Hamiltonian is defined as:

$$H = L + \lambda^t \cdot f \tag{17}$$

Ref.[Speyer] shows that the the second variation of the cost in terms of the Fourier transform of the control is given by:

$$4\pi j\delta^2 J = \int_{-j\infty}^{j\infty} \left[u(-j\omega)^t \, \Pi(\omega) \, u(j\omega) \right] d\omega \tag{18}$$

where,

$$\Pi(\omega) = G^t(-j\omega)H_{zz}G(j\omega) + H_{zu}^t G(j\omega) + G^t(-j\omega)H_{zu} + H_{uu} \tag{19}$$

where

$$G(j\omega) = [j\omega - f_z]^{-1} f_u$$

Here, the matrices f_z, f_u , H_{zz}, H_{zu} and H_{uu} are evaluated at the steady-state condition.

Now, if the (2X2) matrix $\Pi(\omega)$ is nonnegative definite for all frequencies, then $\delta^2 J$ is nonegative Ref[Willems,71]. This is a necessary and sufficient condition. Note that $\delta^2 J$ is nonnegative for <u>any</u> neighboring control (and not just oscillatory variations). Then the steady-state is a local minimum.

On the other hand, if, at some frequency, $\Pi(\omega)$ is not nonnegative definite, an oscillatory variation can be found that makes $\delta^2 J$ negative. This implies that the steady-state solution is not a local minimum. The definiteness of the matrix is easily checked by finding the eigenvalues of Π for all ω. Figure 3 shows the eigenvalues as a function of the frequency for the **cruise** problem, i.e., $\theta = 0$. It is seen that for a range of frequencies, an eigenvalue is negative. This implies that, for the **cruise** problem, the steady-state solution is not a local minimum. This agrees with the results of Ref[Speyer,76] which are obtained for a simpler drag

model. Figure 4 shows how that particular eigenvalue varies with frequency when the cost weighting parameter is increased from 0 to 0.75 N/s. At the higher end, it is seen that the eigenvalue is non-zero for all frequencies. For a simpler drag model, it has been shown in Ref[Karl,87], that for all values of the weighting parameter above a certain value, the steady-state solution passes the frequency test. Thus it can be inferred here that there exists a certain value of $\theta(< 0.75\ N/s)$ above which, the steady-state solution is a weak local minimum for the cruise-dash problem.

For the cases where the steady-state solution is not optimal, an oscillatory variation can be found that gives a lower cost. Going back through the transformations, the variational control history can be found. It is shown in Ref[Speyer76], that these are composed of oscillatory and feedback terms.

It must be noted that these solutions are in the neighborhood of the steady-state solutions. Nothing can be said of solutions far removed from these neighborhoods.

In the next section, periodic solutions in the point-mass modeling are sought.

Point-Mass Periodic Solutions:

At low values of the cost-weighting parameter, theta, frequency-domain analysis indicates that the steady-state solution is not optimal. A control that is periodic about the steady-state with a certain frequency can provide lower costs than the steady-state control. However, at higher values of theta, when the emphasis on time is higher, the steady-state control <u>is</u> locally optimal. Nothing can be said

about the optimality of a solution that is far removed from the steady-state solution. It may be that, even though the steady-state solution is locally optimal at higher values of theta, a periodic control may still beat it. To investigate this, periodic solutions to the cruise-dash problem in the point-mass model are sought here. Rather than construct the periodic solutions suggested by the frequency test (at least at low vales of theta), the cruise-dash problem is formulated as a periodic optimal control problem and the solution obtained from first principles. Periodic solutions to the cruise problem for the aircraft model used here can be found in the literature [Grimm]. The analysis is extended here to include cruise-dash cases. The costs from the resulting solutions can then be compared with those obtained from steady controls.

The point-mass model is used in this formulation. The equations of motion and the boundary conditions for the cruise case can be found in Ref.[Grimm]. Along the same lines, the equations of motion for the cruise-dash problem are derived here.

The objective is to find the controls $C_L(\cdot)$ and $\eta(\cdot)$ that minimize the cost functional:

$$J = \frac{1}{X_f} \int_0^{X_f} \frac{\eta Q + \theta}{V \cos \gamma} dX \tag{20}$$

The states are the velocity (V), the path-angle (γ) and the altitude (h). The states are governed by the differential equations given in chapter two:

$$V' = \frac{g}{V\cos\gamma}\left[\frac{\eta T - D}{W} - \sin\gamma\right]$$

$$\gamma' = \frac{g}{V^2}\left[\frac{L}{W\cos\gamma} - 1\right] \tag{21}$$

$$h' = \tan\gamma$$

The available controls are the lift-coefficient (C_L) and the throttle (η). The states satisfy the periodic boundary conditions:

$$\begin{aligned} V(0) &= V(X_f) \\ \gamma(0) &= \gamma(X_f) \\ h(0) &= h(X_f) \end{aligned} \tag{22}$$

The wavelength, X_f, of the oscillations is unknown and has to be determined as part of the control problem.

The variational Hamiltonian for the problem is defined in the usual way:

$$\begin{aligned} H = \frac{\eta Q + \theta}{V\cos\gamma} + \lambda_V \frac{g}{V\cos\gamma}[P - \sin\gamma] + \\ \lambda_\gamma \frac{g}{V^2}\left[\frac{L}{W\cos\gamma} - 1\right] + \lambda_h \tan\gamma \end{aligned} \tag{23}$$

The independent variable, X, does not appear in the right-hand sides of the equations or in the integrand of the cost functional. Hence, the Hamiltonian is a constant of the system.

Setting the first variation of the cost to zero [Bryson] gives the governing equations for the co-states:

$$\lambda_V' = -\frac{\partial H}{\partial V}$$

$$\lambda_\gamma' = -\frac{\partial H}{\partial \gamma} \tag{24}$$

$$\lambda_h' = -\frac{\partial H}{\partial h}$$

The analytical expressions for $\frac{\partial H}{\partial V}$, $\frac{\partial H}{\partial \gamma}$, and $\frac{\partial H}{\partial h}$ are given in Ref.[Thesis]. The first variation condition also specifies the transversality conditions:

$$\begin{aligned} \lambda_V(0) &= \lambda_V(X_f) \\ \lambda_\gamma(0) &= \lambda_\gamma(X_f) \\ \lambda_h(0) &= \lambda_h(X_f) \end{aligned} \tag{25}$$

The boundary condition that determines the free wavelength is [Evans]:

$$H = J = \frac{1}{X_f} \int_0^{X_f} \frac{\eta Q + \theta}{V \cos \gamma} dX \tag{26}$$

The rank of the periodic boundary conditions (equations (22) and (25)) is one less than full, due to the periodic nature of the solutions. This is because the boundary conditions can be satisfied any of a one-parameter family of solutions. One member of the family has to be "tied" down. This can be accomplished by

specifying the actual initial (or final) value of one of the variables (within their limits). The following set of boundary conditions does just that:

$$\begin{aligned} V(0) &= V(X_f) \\ \gamma(0) &= 0 \\ h(0) &= h(X_f) \\ \lambda_V(0) &= \lambda_V(X_f) \\ \lambda_\gamma(0) &= \lambda_\gamma(X_f) \\ \lambda_h(0) &= \lambda_h(X_f) \end{aligned} \tag{27}$$

The fact that the Hamiltonian is the same on both ends (it is a constant) ensures that the path-angle is zero at the other end (to the order of accuracy of the numerical method).

The Pontryagin minimum principle determines the optimal controls. The lift-coefficient appears nonlinearly in the Hamiltonian and so the optimal lift-coefficient is given by:

$$\frac{\partial H}{\partial C_L} = 0 \rightarrow \lambda_\gamma - \lambda_V V \frac{\partial C_D}{\partial C_L} = 0 \tag{28}$$

with the second order sufficient condition:

$$\frac{\partial^2 H}{\partial {C_L}^2} = -\lambda_V V \frac{\partial^2 C_D}{\partial {C_L}^2} > 0 \tag{29}$$

The Hamiltonian is linear in the throttle which makes this control bang-bang or singular. The optimal throttle is given by:

$$\eta = 0 \quad \text{for} \quad S > 0$$
$$= 1 \quad \text{for} \quad S < 0 \tag{30}$$
$$\text{from } \ddot{S} = 0 \quad \text{for} \quad S \equiv 0$$

Here, S is the throttle switching function $\frac{\partial H}{\partial \eta}$. As the solutions are periodic, one expects an even number of switches in the trajectory. $S \equiv 0$ is the condition for a singular arc [Bryson].

Numerical Solutions:

The problem is obviously too complex for analytical solutions. So numerical solutions are sought. Ref[Grimm] uses the multiple shooting algorithm [Bulirsch] to find the numerical solutions for the cruise problem. The same algorithm was used in this effort. With the solution of [Grimm] as a starting point, continuation methods were used to find the solutions for the cruise-dash case ($\theta > 0$). The solutions were found for theta ranging from 0 to 0.21 N/s.

Ref [Grimm] shows that the cruise problem does not have a singular arc in the solution. Based on this, singular arcs are not expected in the cruise-dash solutions. Two switching points were assumed. However, the switching function was closely monitored along the whole trajectory. Figure 5 shows the switching function against the range normalized by the wavelength for theta varying from 0 to 0.21 N/s. This is an a posteriori justification of the two-switching-point assumption. However, for the case θ = 0.21 N/s, the switching function has a noticeably small slope at the second switching point. This does not augur well for the bang-bang control structure, but suggests a more interesting, though complex, bang-singular arc combination to challenge the (now) garden-variety bang-bang periodic sol-

utions. Figure 6 confirms the best of hopes (or the worst fears). This is a candidate solution for the case $\theta = 0.25$ N/s. This was obtained with the bang-bang assumption. The switching function actually crosses the zero axis at a few closely spaced points. This is a suggestive indication of the presence of a singular arc. However, interesting as the problem is, the solution is very complex. As the aircraft model is very intricate, the derivation of the singular control laws is very difficult, but not out of reach. Unfortunately, due to time pressures, that part of the investigation has not been carried out. The good news is that it provides an opportunity for future effort. Breakwell [Breakwell,87] has used a quadratic approximation analysis that seems to work amazingly well in predicting the point-mass periodic solution for the cruise case. This could be the road to take in trying to find the bang-singular solutions.

Figure 7 shows the periodic solutions for the cruise-dash problem. Theta varies from 0 to 0.21 N/s. Shown in that Figure is a cross-plot of the velocity and altitude trajectories. The corners in the trajectories are the switching points where the throttle jumps. At the lower altitude switching point, the throttle jumps from full to zero. At the other corner, the throttle jumps from 0 to 1.

Figure 8 through Figure 13 show the states and co-states as functions of the range normalized by the wavelength. This is shown for several values of theta ranging from 0 to 0.21 N/s. Figures 8, 9, and 10 show the states, in order, velocity, path-angle and altitude. Figures 11, 12 and 13 show their respective co-states. Figure 14 shows the lift-coefficient versus the normalized time.

Figure 15 shows the costs for periodic controls as the cost-weighting parameter theta is varied. Also shown, for comparison, are the costs for the steady-state solutions. For better visual comparison, figure 16 shows the difference in those costs versus theta, expressed as a difference percentage over the steady-state costs. It is seen for lower values of theta (including zero, verifying Ref[Grimm]), the periodic control yields lower costs than the singular arc. However, as theta increases, the gain decreases. Also, over the whole range of theta, the costs do no differ by much. At $\theta = 0$, the periodic solutions are lower than the steady-state costs by about two percent. At $\theta = 0.21$ N/s, the periodic solutions are about half a percent lower.

However, this is not the complete picture. The real performance criterion is the costs against average velocity. Figure 17 shows the two costs against average velocity achieved. This shows that the disparity between the costs is higher. However, the difference is not very much even at lower average velocities. The question whether the cost improvement is worth the complexity of the periodic solutions is somewhat subjective and cannot be easily answered. An important point to remember in this connection is that the steady-state solutions are locally optimal at higher values of theta.

Based on figures 5,6 and 15-17, one can conjecture the form of the solutions for values of theta higher than 0.21 N/s. It seems reasonable to assume that part of the periodic trajectory is a singular arc. In figure 5, the amplitude of the switching function is seen to be decreasing as the weight on the time is made higher. The periodic trajectory costs seem to approach the steady-state costs (Figures 45-47). Figure 7 shows that the amplitudes of the states keep decreasing. It is well known

that the optimal solution for the dash problem ($\theta \rightarrow \infty$) is full throttle over the whole trajectory. All this would *seem* to indicate that as the weight on time is increased, the singular portion of the trajectory would increase, until the entire solution is a singular arc (degenerating into the steady-state solution). In this connection, it would not be amiss to mention that after some value of theta, the steady-state solution is locally optimal. Ref.[Karl] has some preliminary results for the related problem of approaching the steady-state solution from arbitrary initial states in the neighborhood. This is a good first step in the investigation of the bang-singular type of periodic solutions.

<u>Optimality of the Periodic Solutions</u>:

From the analysis conducted so far, a few general remarks can be made about the optimality of the point-mass solutions.

First, for high values of theta, the steady-state solutions are locally optimal. For lower values of theta, they are <u>not</u> optimal. So at some value of theta between 0 and 0.21 N/s, the steady-state solution definitely has a "conjugate" point. Indeed, there may even be a "Darboux" point [1][Ref ?], <u>if</u> it the steady-state is globally optimal at higher values of theta.

Second, for low values of theta, the periodic solutions are locally optimal (Ref[Grimm] has shown this to be true). The steady-state solutions are not even locally optimal. The periodic solutions at this end <u>may</u> be globally optimal. At the zeroth order approximation, the relaxation oscillations have lower costs. No pre-

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[1] A solution loses its global optimality at a Darboux point.

diction can be made about higher approximations. So the point-mass periodic solutions may or may not have a contender. The optimality the periodic solutions at higher values of theta can be considered only after finding the solutions. The difficulty in determining the optimality is several orders more complex than finding the solutions. It is only recently that a Jacobi-type test has been formulated for the bang-bang type of periodic solutions [Evans].

To make the third point, the frequency of oscillation for the cruise problem is indicated by a vertical line in Figure 2. It is very interesting that the frequency of oscillations are in the range suggested by the frequency domain analysis of the This seems to indicate that the control constructed from the frequency analysis (Ref[Speyer,1976] describes how they are obtained) is a good guess for the point-mass solutions. If so, it would indicate that for high enough values of theta, the steady-state is ***the*** periodic solution.

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Figure 1. Point Mass Steady State Solution: Cost vs. Theta

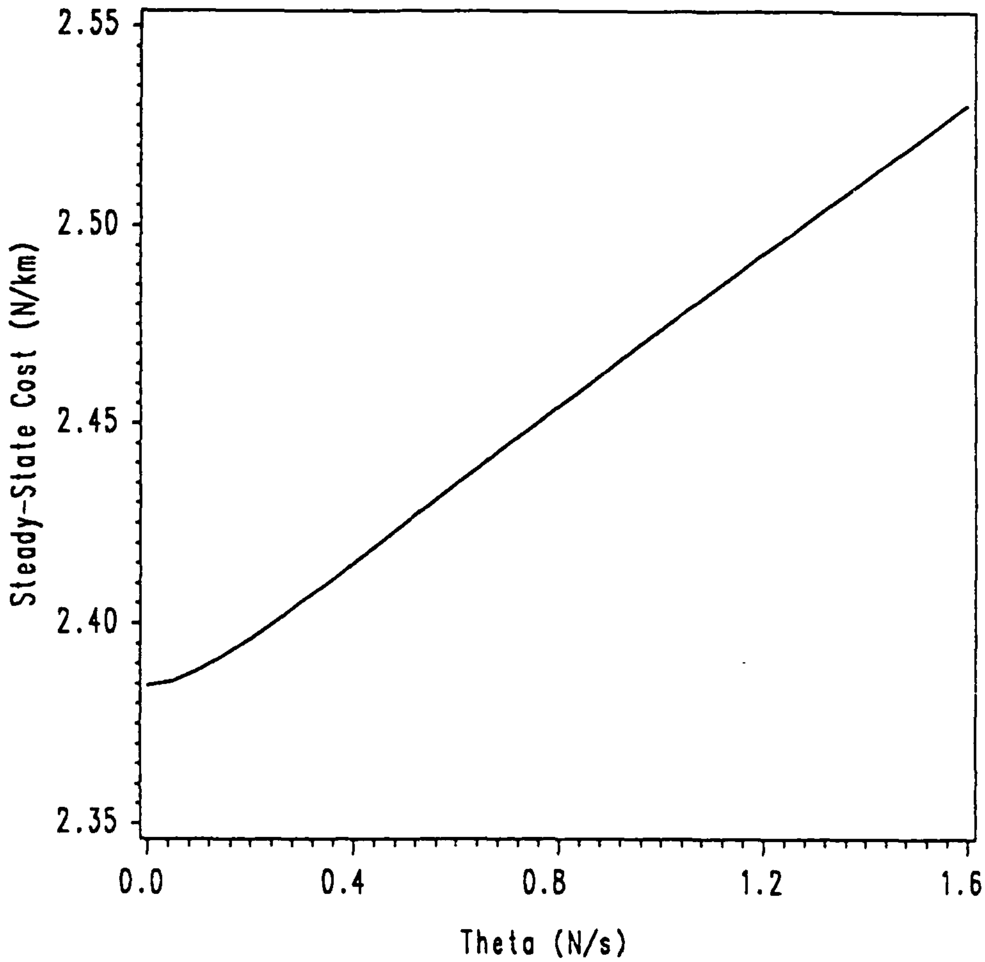

Figure 1. Point Mass Steady State Solution: Cost vs. Theta

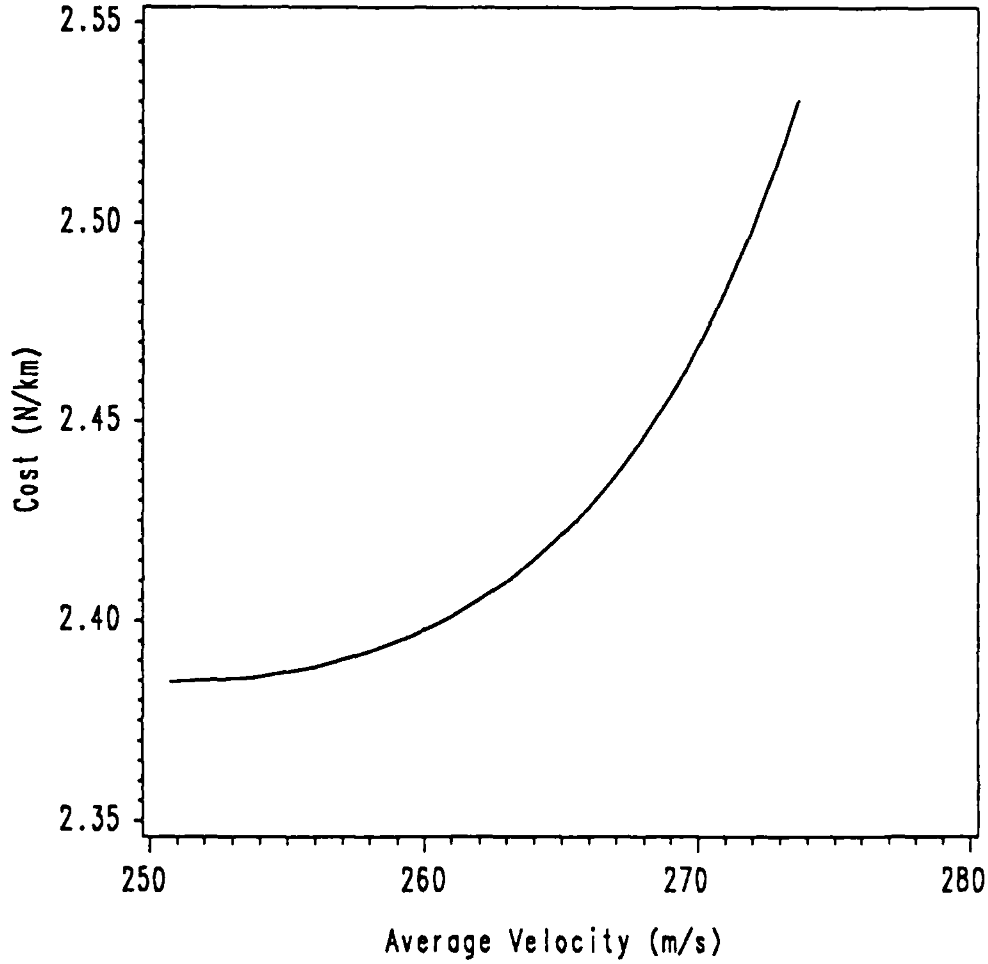

Figure 2. Point Mass Steady State Solution: Cost vs. Average Velocity

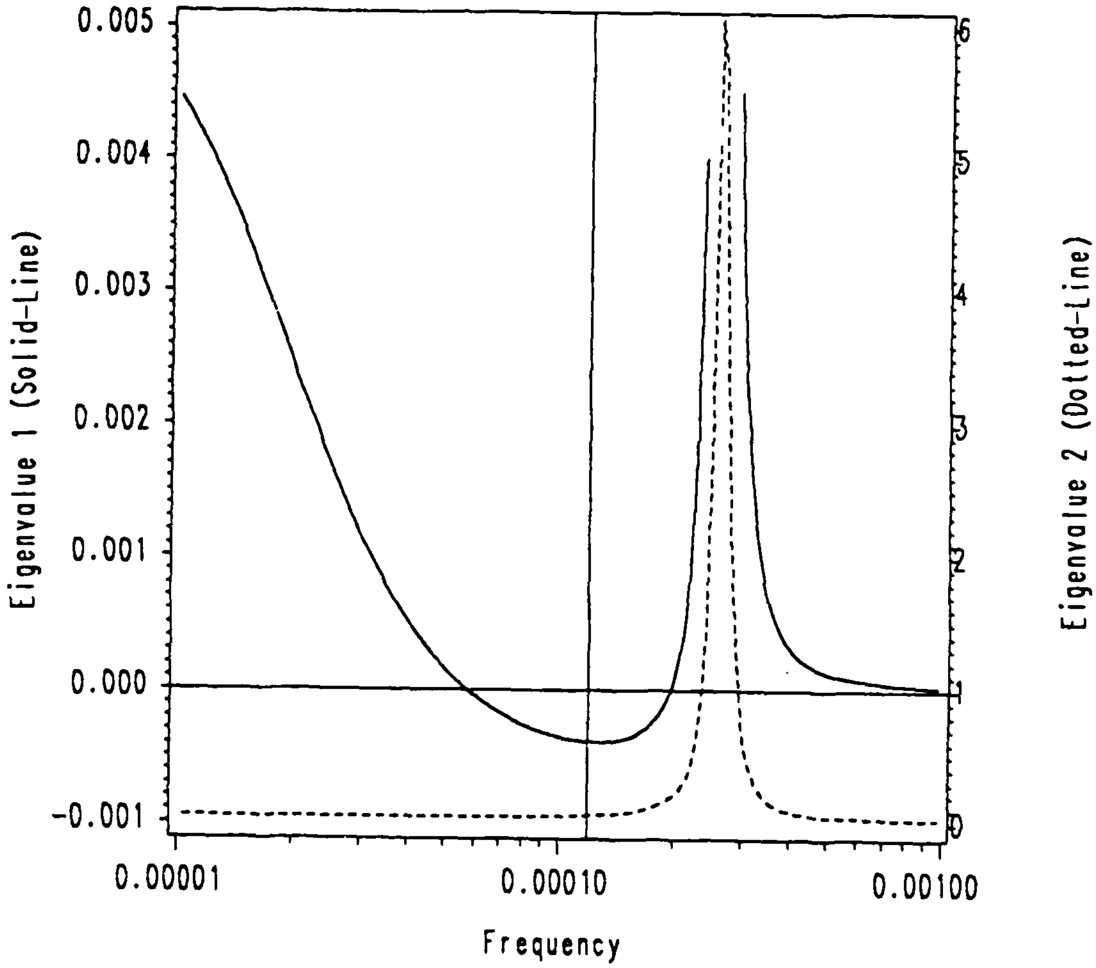

Figure 3. The Frequency Test: Eigenvalues vs. Frequency ($\theta = 0$)

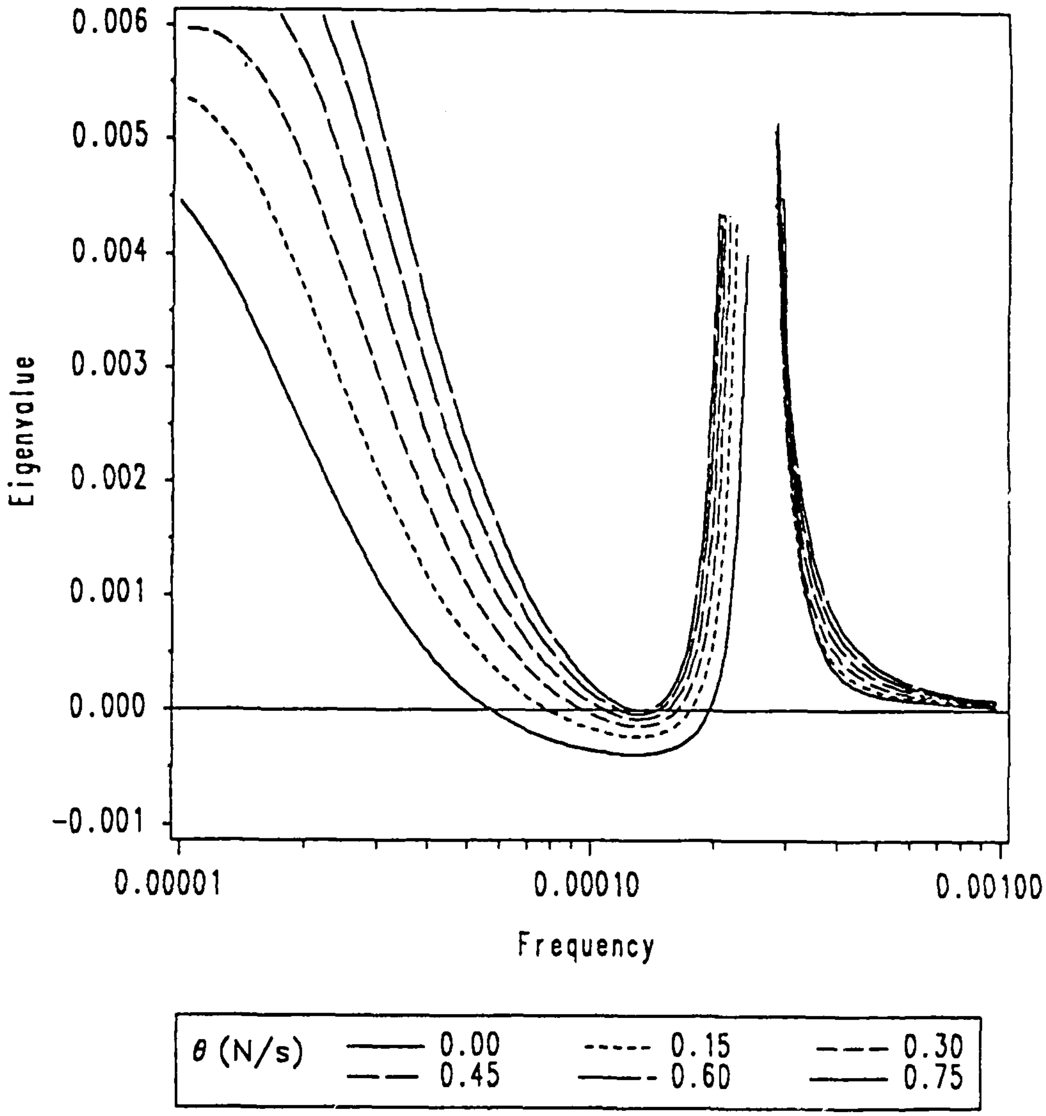

Figure 4. The Frequency Test: The Errant Eigenvalue vs. Frequency

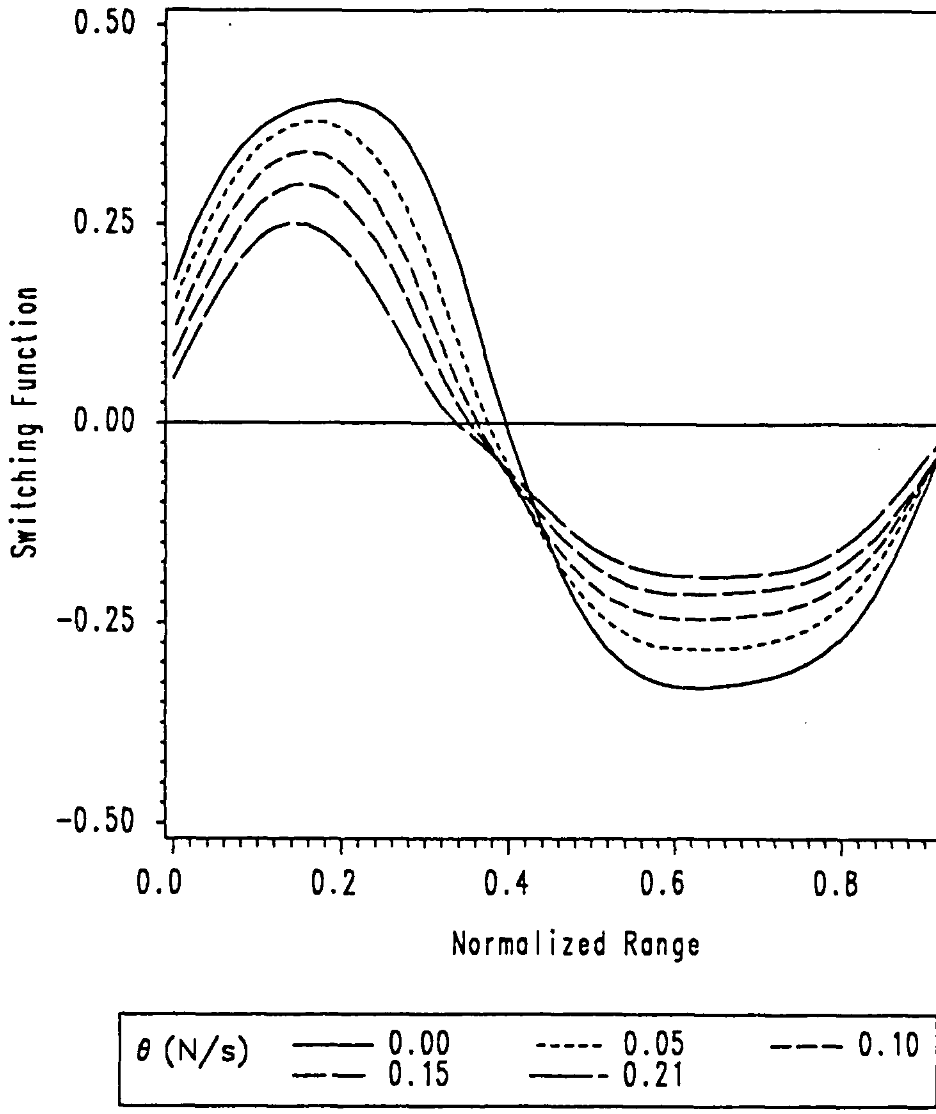

Figure 5. Point-Mass Periodic Solutions: Switching Function History

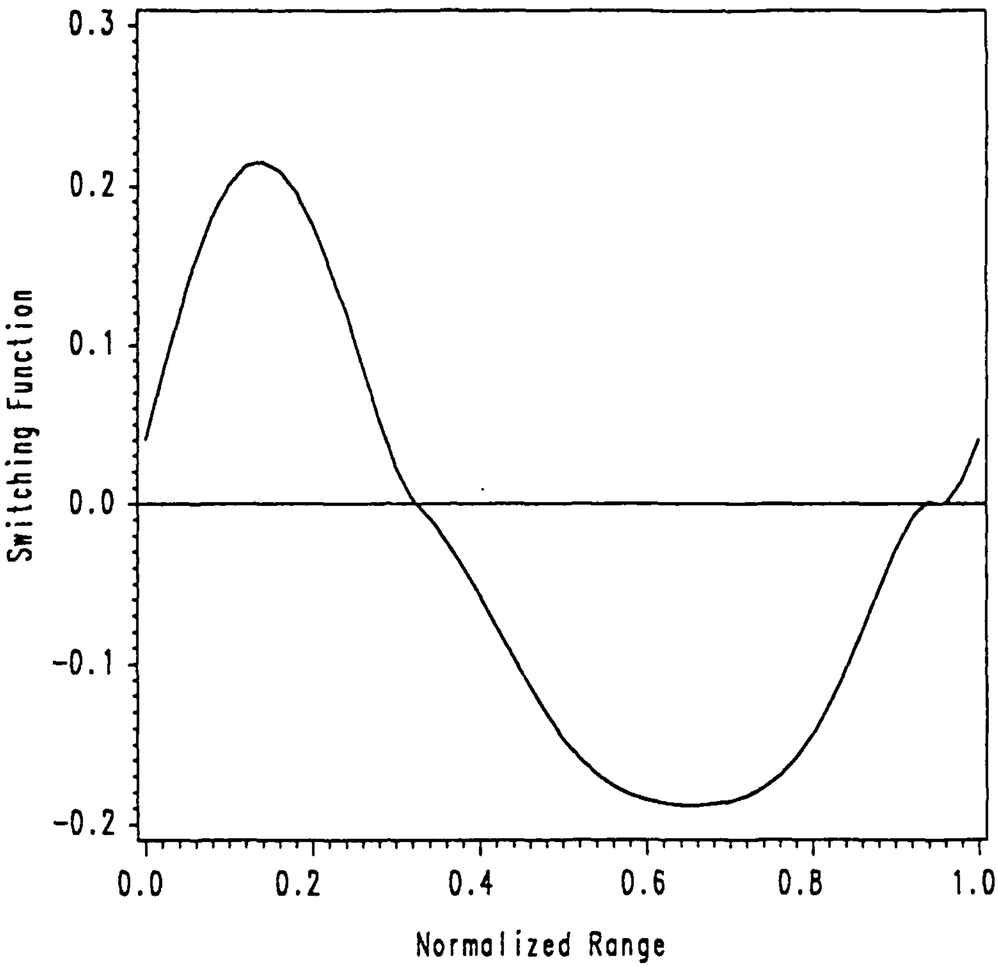

Figure 6. Switching Function History for the Case $\theta = 0.25\ N/s$

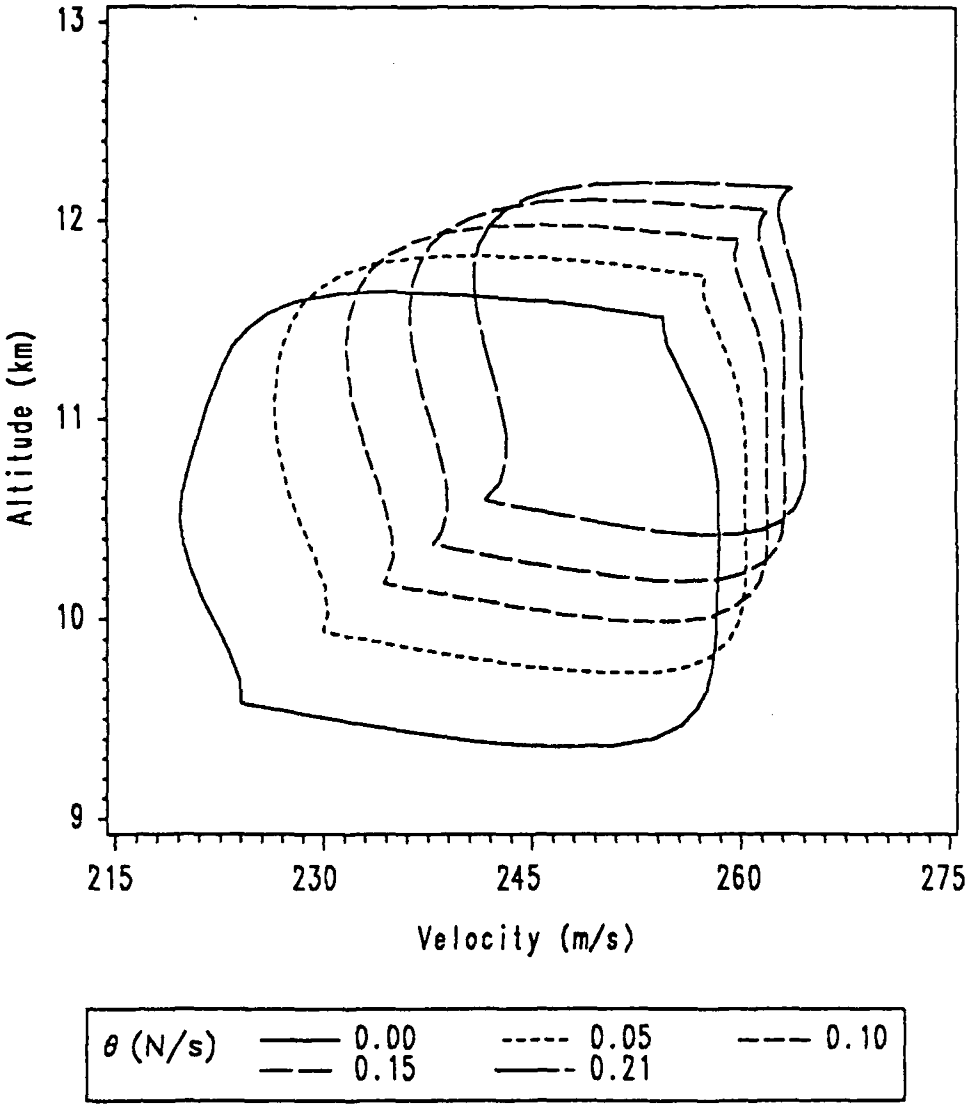

Figure 7. Point-Mass Periodic Solutions: Altitude vs. Velocity

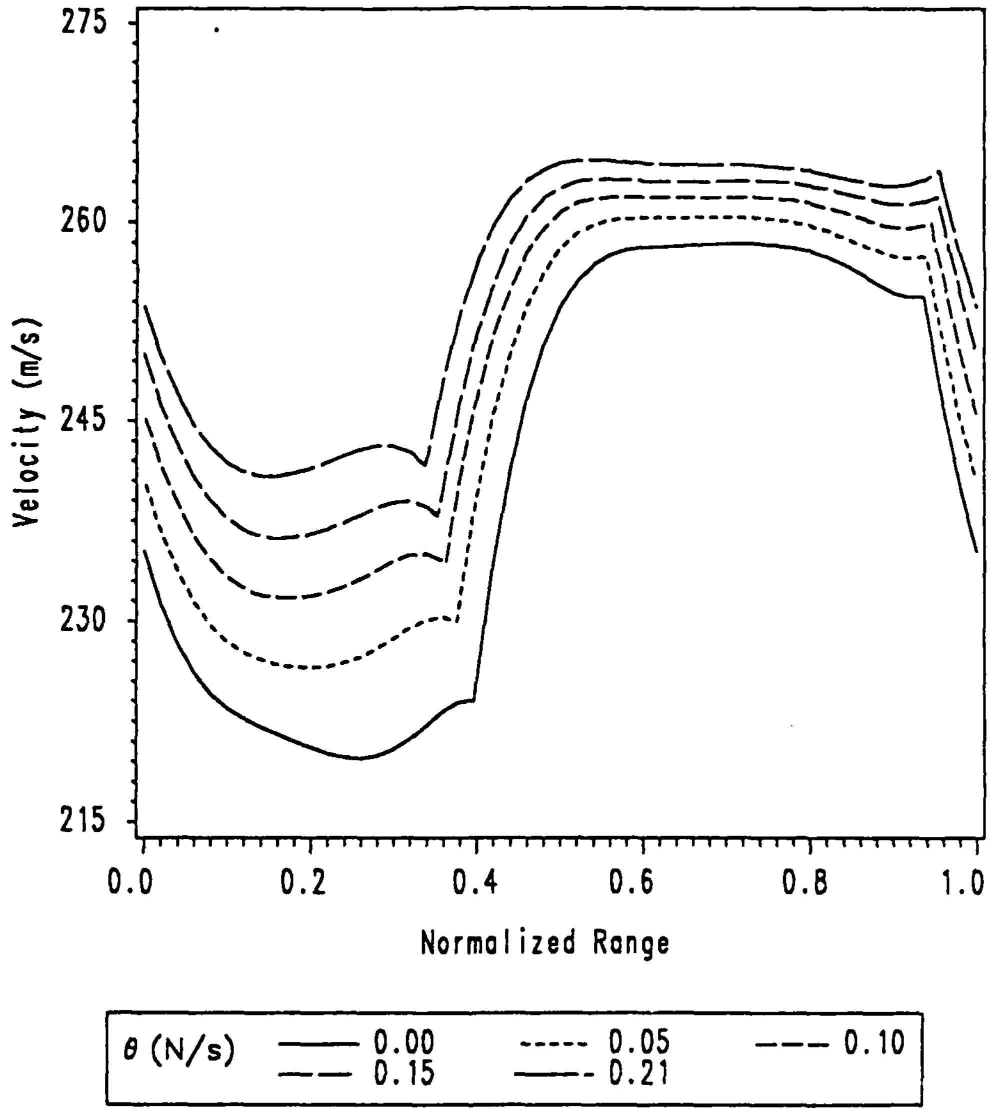

Figure 8. Point-Mass Periodic Solutions: Velocity History

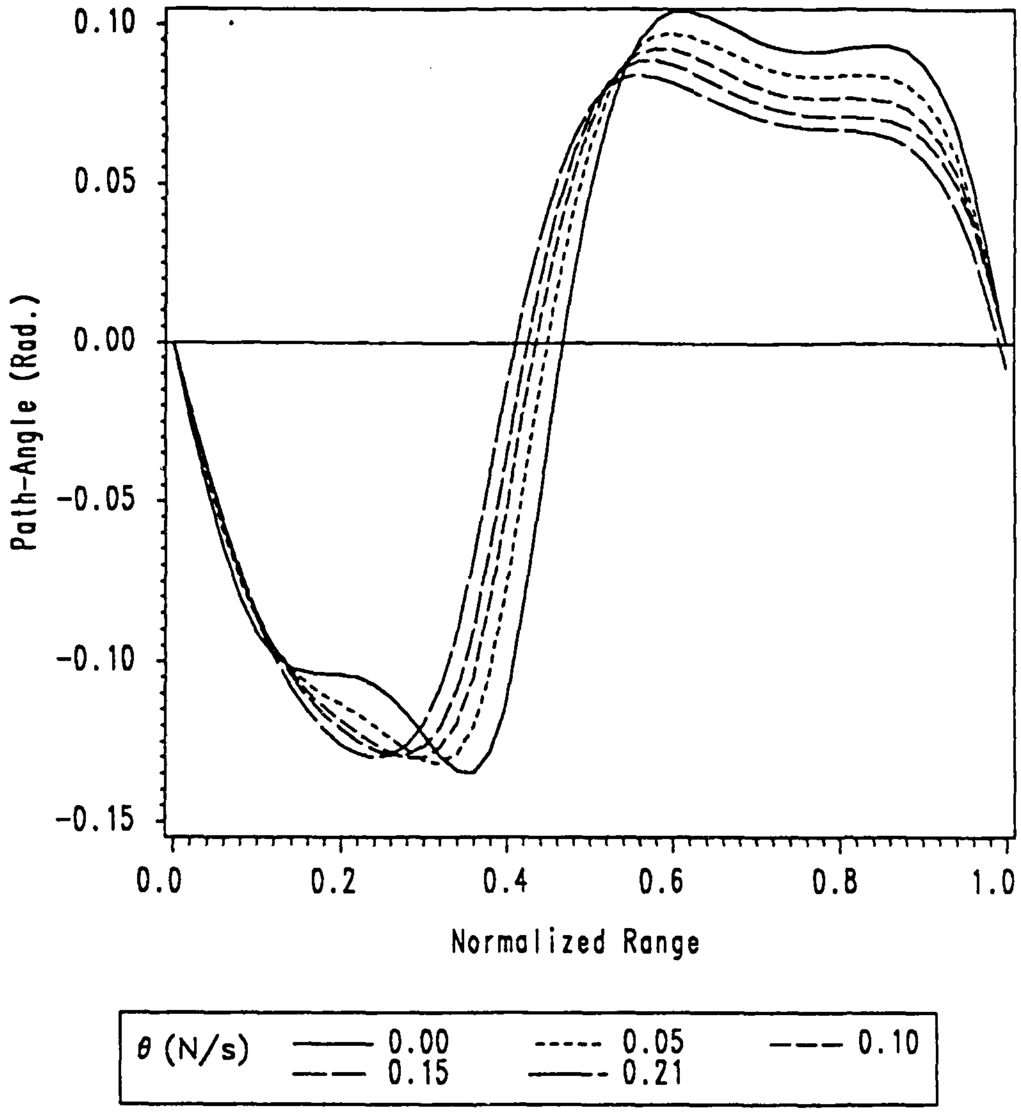

Figure 9. Point-Mass Periodic Solutions: Path-Angle History

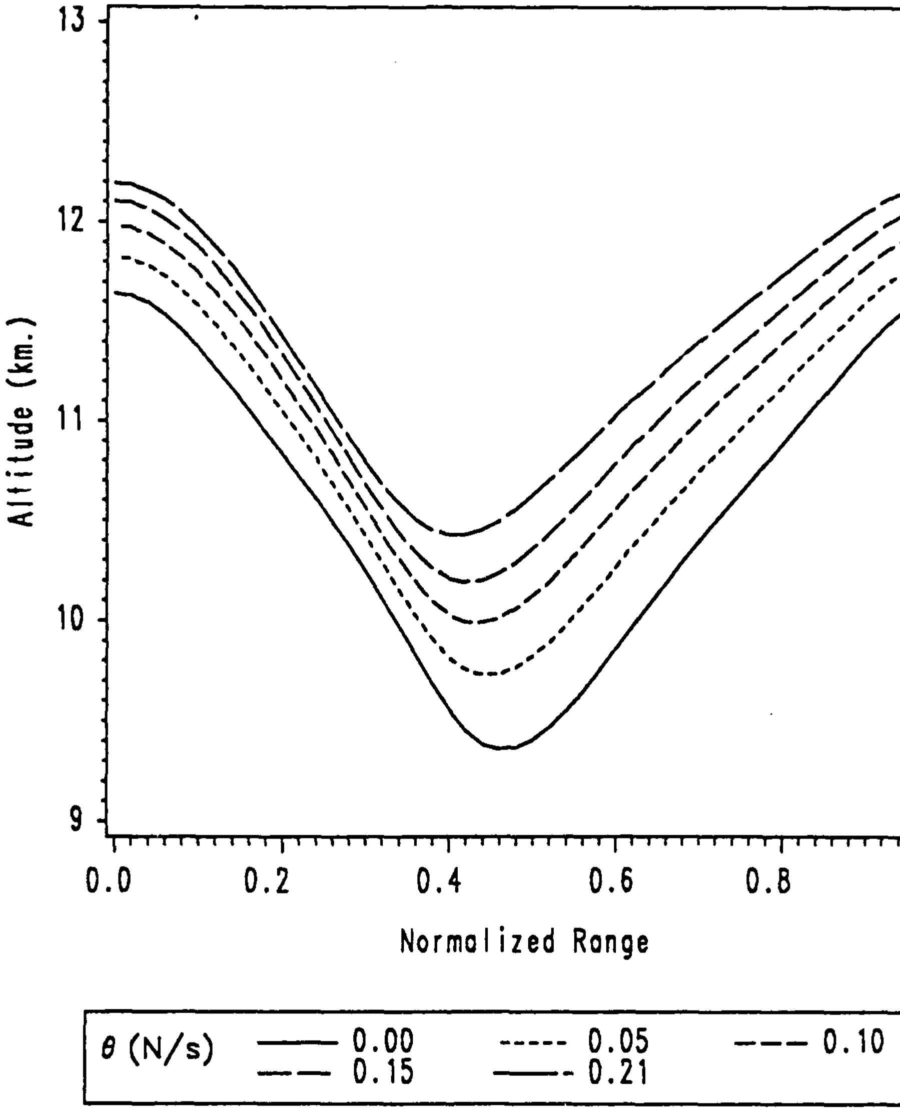

Figure 10. Point-Mass Periodic Solutions: Altitude History

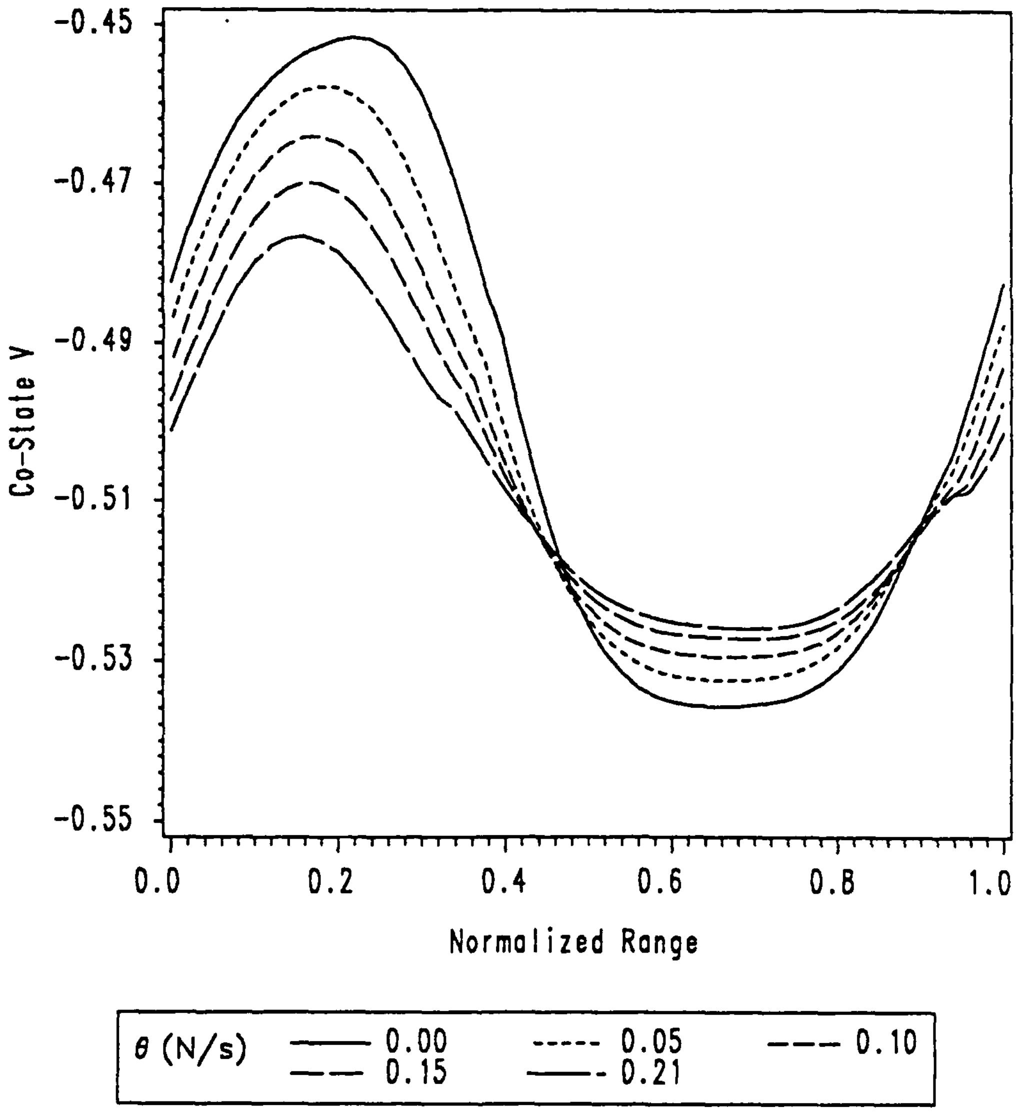

Figure 11. Point-Mass Periodic Solutions: Costate-V History

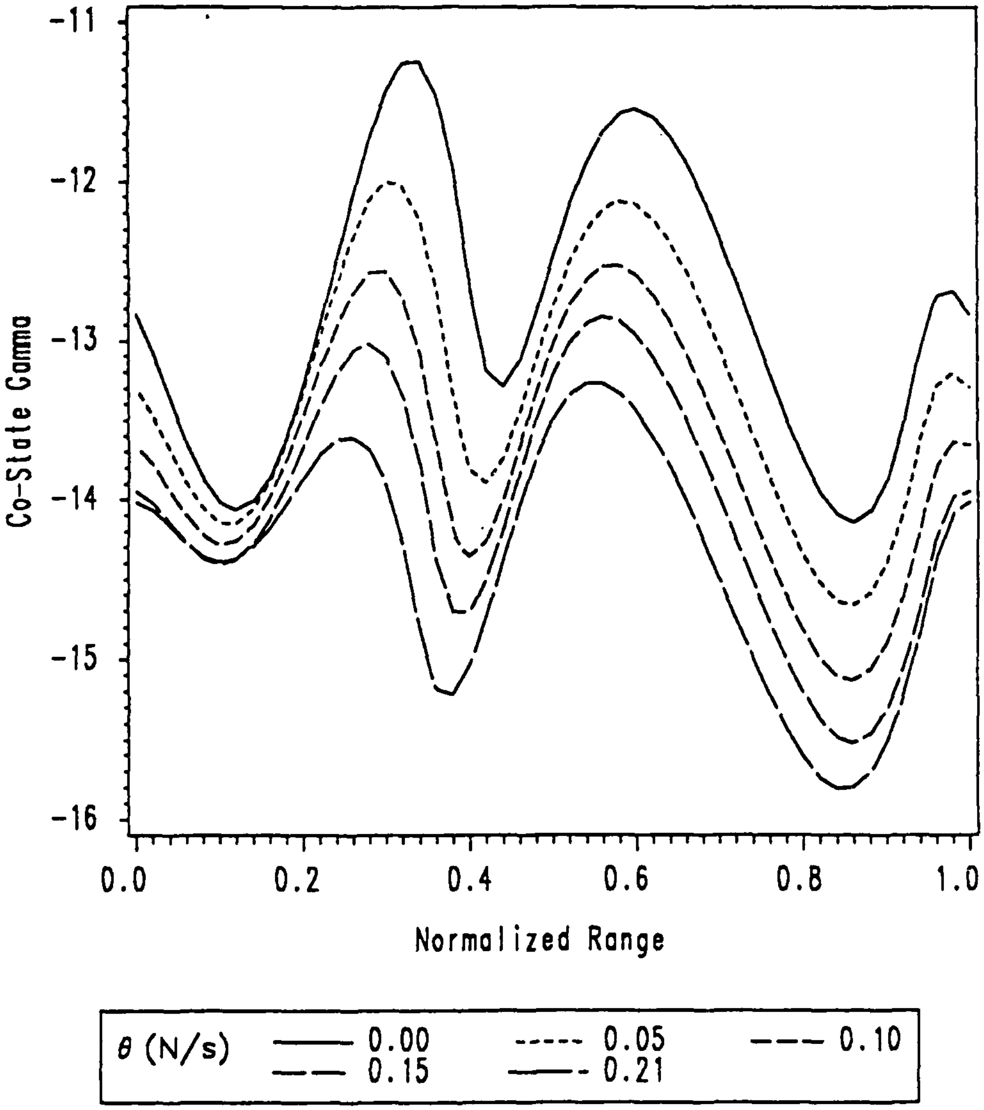

Figure 12. Point-Mass Periodic Solutions: Costate-γ History

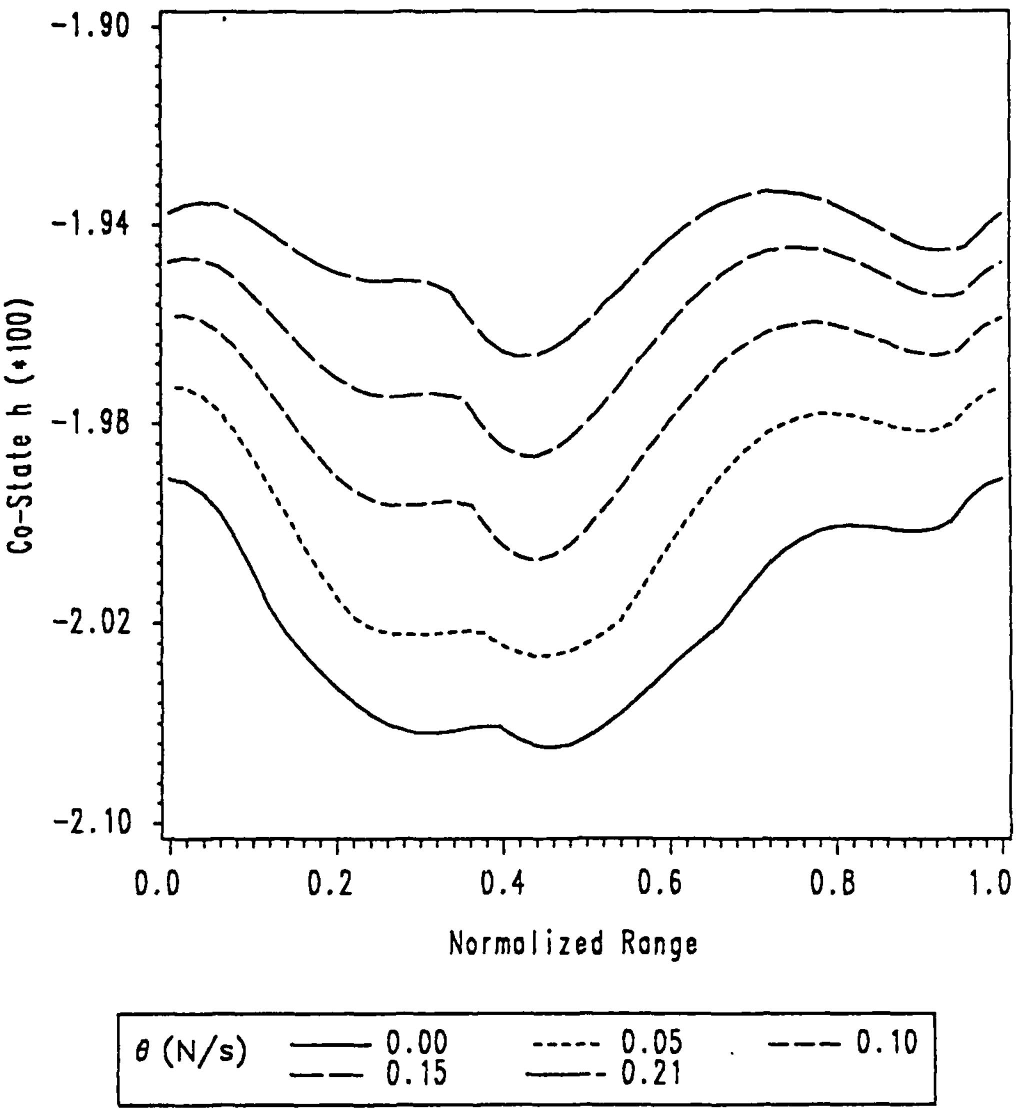

Figure 13. Point-Mass Periodic Solutions: Costate-h History

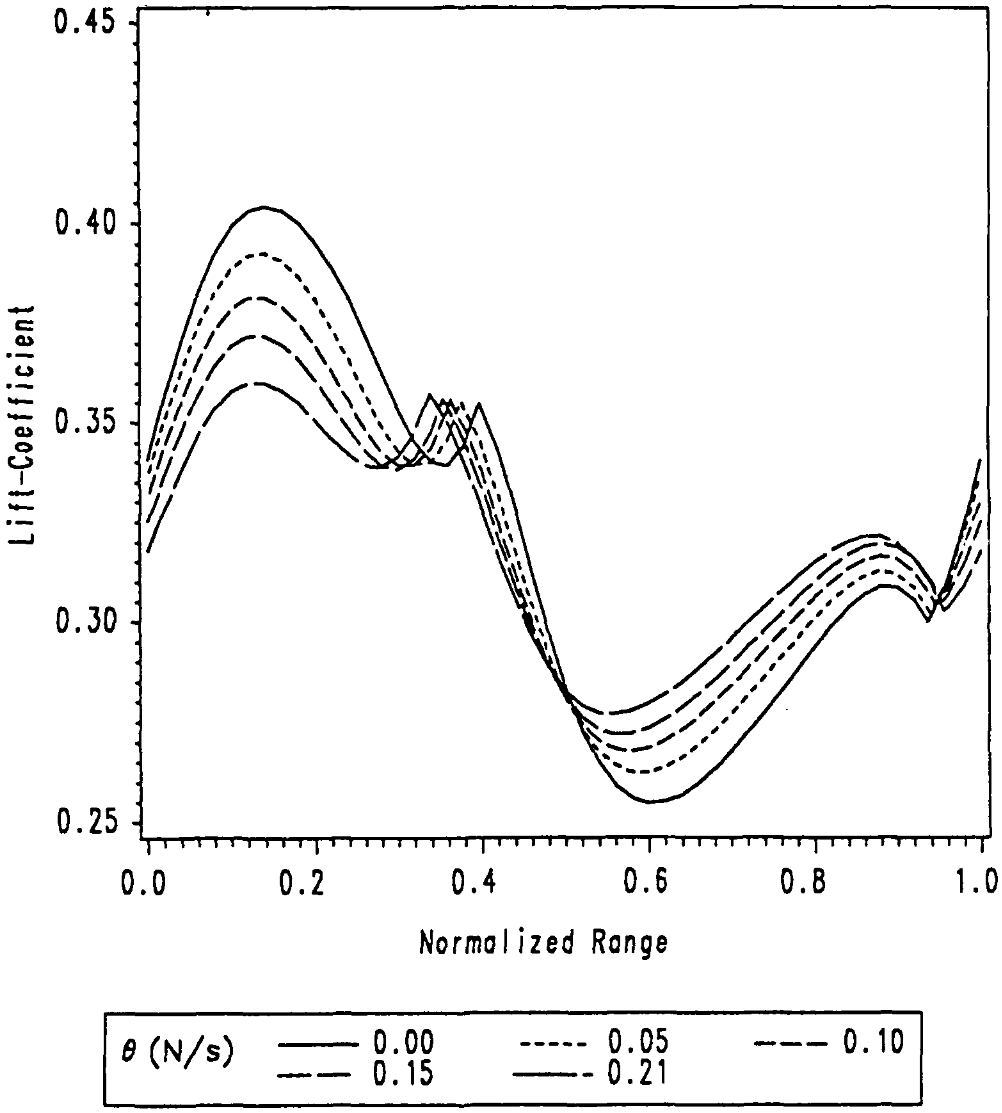

Figure 14. Point-Mass Periodic Solutions: Lift-Coefficient History

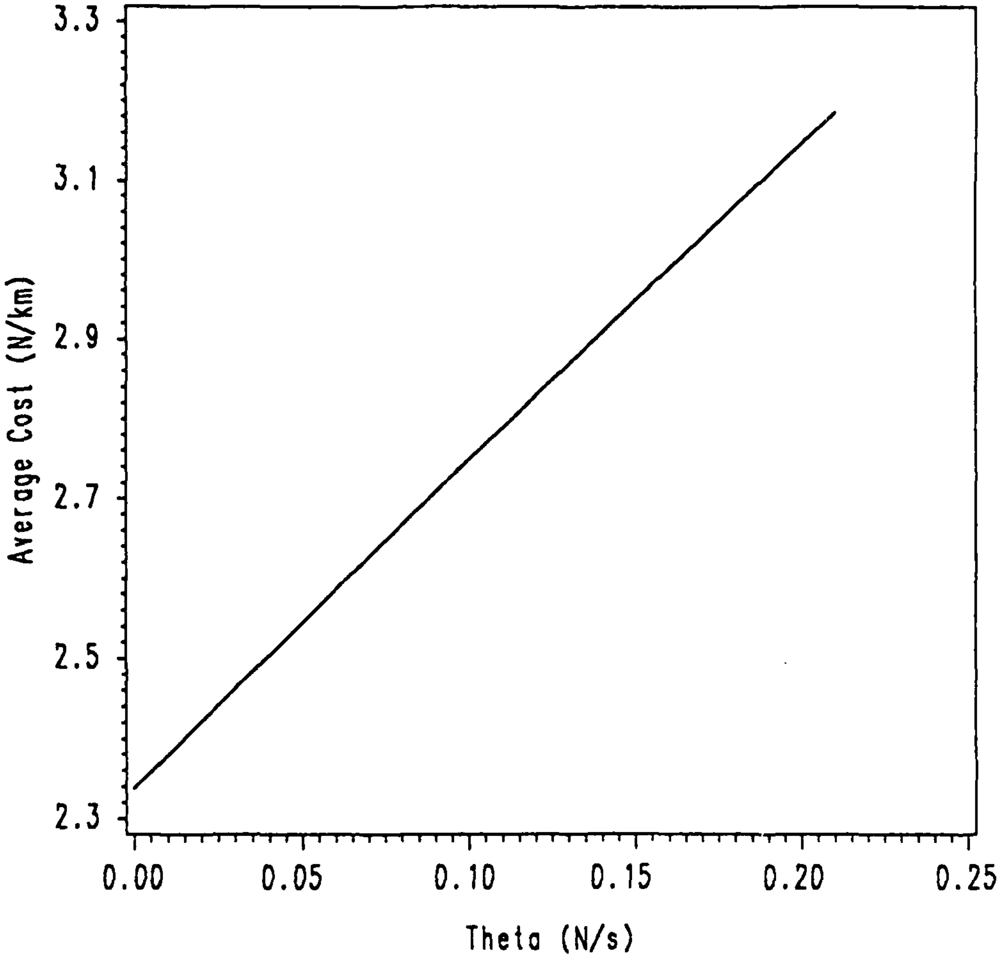

Figure 15. Point-Mass Periodic Solutions: Cost vs. Theta.

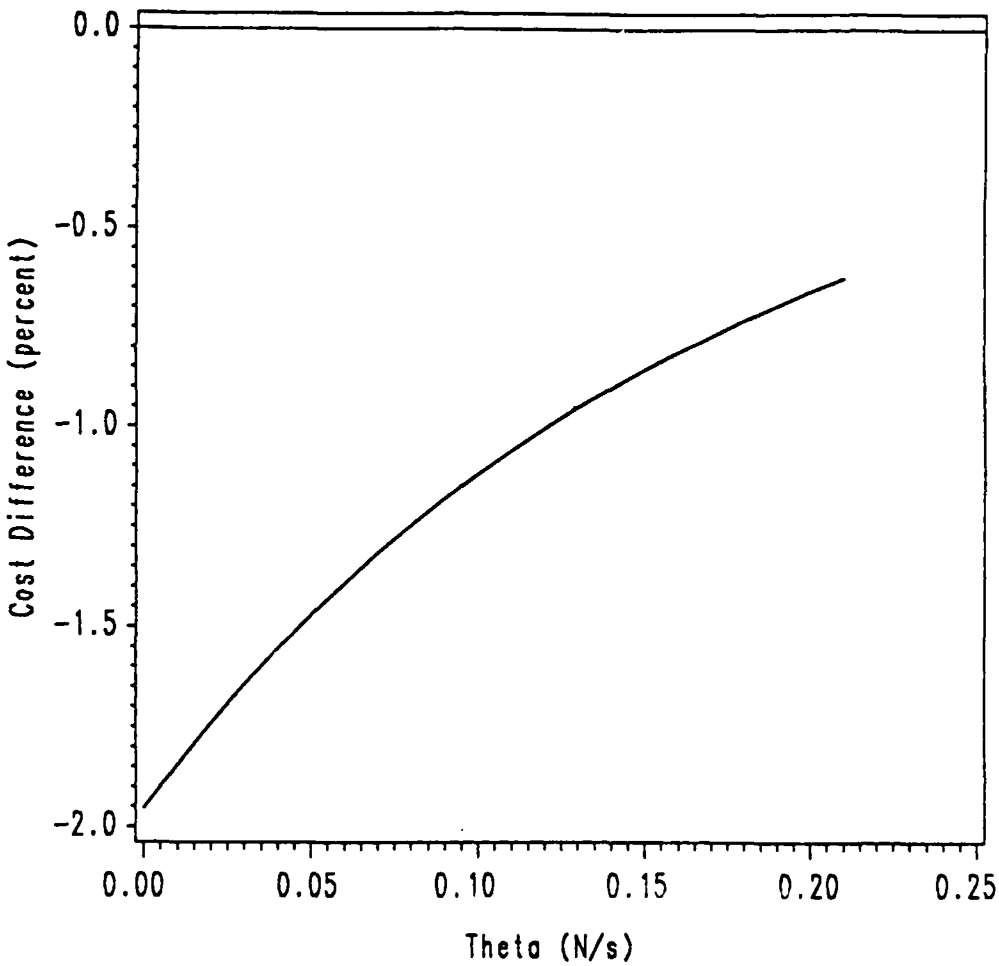

Figure 16. Periodic and Steady-State Solutions
Cost Difference (percent) vs. Theta.

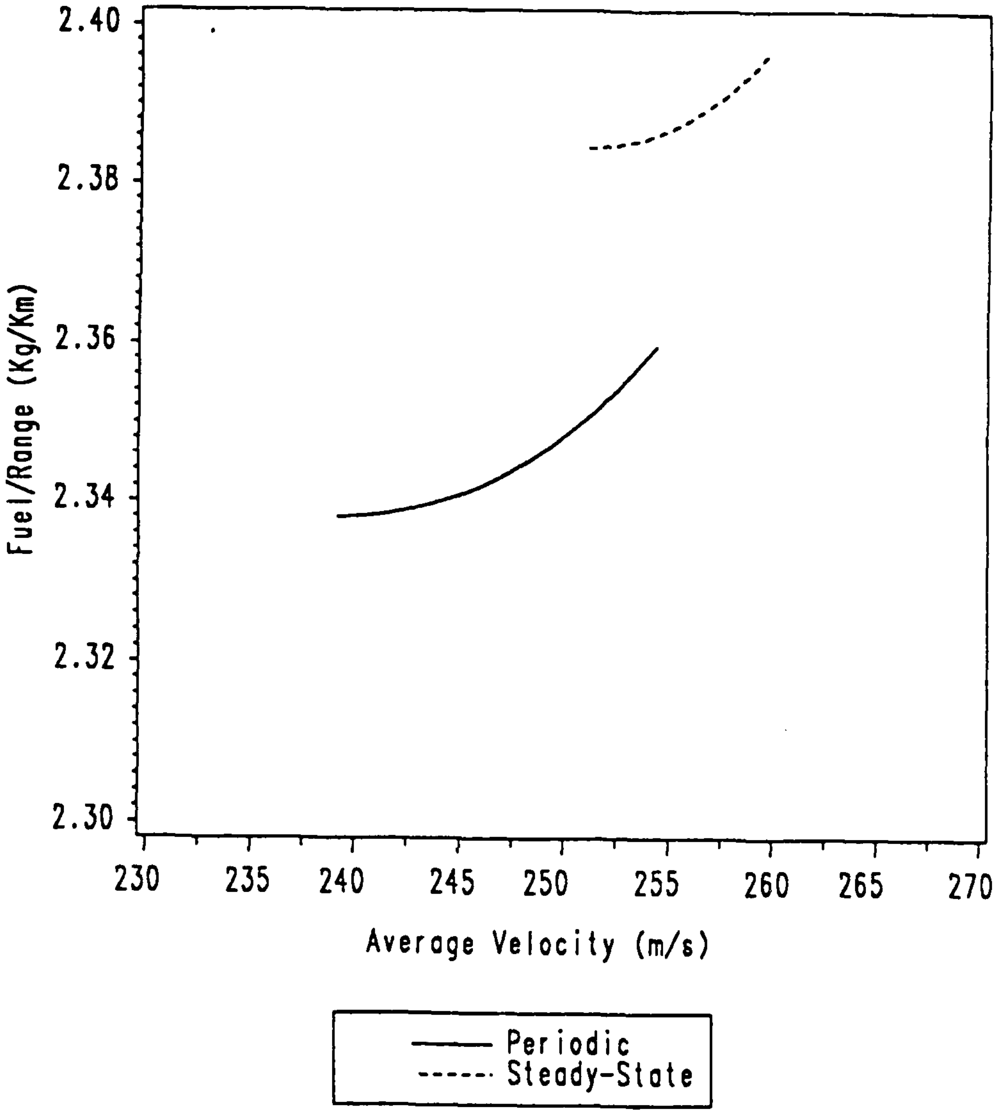

Figure 17. Periodic and Steady-State Solutions
Costs versus Average Velocity